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A STABILITY PROPERTY OF THE UNIT VECTOR BASIS OF l_p

BY

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ABSTRACT

Let $\{e_n\}$ be the unit vector basis of l_p, $1 < p < \infty$, and let $x_n = a_n e_n - b_n e_{n+1}$. Necessary and sufficient conditions are given for the operator $T: l_p \to \operatorname{span}\{x_n\}$ defined by $Te_i = x_i$ to be invertible.

1. Introduction

In this paper we prove a stability theorem for the unit vector basis of l_p, $1 < p < \infty$. Two sequences $\{x_n\}$ and $\{y_n\}$ of elements of a Banach space X are called *equivalent* if for every sequence $\{a_n\}$ of scalars, $\Sigma a_n x_n$ converges if and only if $\Sigma a_n y_n$ is convergent. Stability theorems for bases in Banach spaces usually state that if $\{e_n\}$ is a basis and $\{x_n\}$ is "not far" from $\{e_n\}$, then $\{x_n\}$ is equivalent to $\{e_n\}$ or has properties similar to those of $\{e_n\}$. However, in most cases (cf. [1], th. 1) "not far" means that, in some uniform way, the elements of $\{x_n\}$ are close to those of $\{e_n\}$ in norm. In contrast to these classical stability properties we will discuss here a stability property where "not far" has a combinatorial meaning rather than a geometric one.

Let $\{e_n\}$ denote the unit vector basis of l_p, $1 < p < \infty$. A sequence $\{x_n\}$ in l_p is called *almost diagonal* if for each $n \geqq 1$ $x_n = a_n e_n - b_n e_{n+1}$. We call $\{x_n\}$ *semi-normalized* if $0 < \inf_n \|x_n\| \leqq \sup_n \|x_n\| < \infty$. Let $\{x_n\}$ be a sequence of elements of a Banach space X and let $\{f_n\} \subset X^*$. The system $\{x_n, f_n\}$ is called a bounded biorthogonal system if $f_i(x_j) = \delta_{i,j}$ and both $\{x_n\}$ and $\{f_n\}$ are semi-normalized.

Our main result is the following:

THEOREM. *Let $\{x_n\}$ be a semi-normalized almost diagonal system in l_p, $1 < p < \infty$. Then the following statements are equivalent*:

(a) *there is a sequence $\{f_n\} \subset l_p^*$ such that $\{x_n, f_n\}$ is a bounded biorthogonal system.*

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(b) $\{x_n\}$ *forms a basis for* $\operatorname{span}\{x_n\}$.

(c) $\{x_n\}$ *is equivalent to the unit vector basis* $\{e_n\}$. ($\operatorname{span}\{x_n\}$ denotes the closed linear subspace spanned by $\{x_n\}$).

We conclude the introduction with the following three remarks:

REMARK 1. Let $\{x_n\}$ and $\{y_n\}$ be bases of the Banach spaces X and Y respectively. Then $\{x_n\}$ and $\{y_n\}$ are equivalent if and only if the map $T: X \to Y$ defined by $T(\Sigma t_i x_i) = \Sigma t_i y_i$ is an isomorphism of X onto Y.

REMARK 2. The Theorem is false for $p = 1$. Indeed, let $x_n = e_n - e_{n+1}$, let $\{\phi_n\}$ be the unit vector basis of c_0 and put $f_n = \Sigma_{i=1}^n \phi_i$.

Then $\|x_n\| = 2$ and $\|f_n\| = 1$ for all n and $f_i(x_j) = \delta_{i,j}$. Hence $\{x_n, f_n\}$ forms a bounded biorthogonal system and it is easy to check that $\{x_n\}$ is a basis for $\operatorname{span}\{x_n\}$. However, the map T defined by $Te_i = x_i$ is not an isomorphism, because $\|\Sigma_{i=1}^n x_i\| = 2$ and $\|\Sigma_{i=1}^n e_i\| = n$ for all n.

REMARK 3. The Theorem is trivially true for c_0. Indeed, let $\sup\{|a_n|, |b_n|\} = M < \infty$, then $\|\Sigma t_i x_i\| \leqq \|\Sigma a_i t_i e_i\| + \|\Sigma b_i t_i e_{i+1}\| \leqq 2M \|\Sigma t_i e_i\|$ and hence $\Sigma t_i x_i$ converges whenever $t_i \to 0$. On the other hand, if $\Sigma t_i x_i$ converges then $t_i \to 0$ because $\inf_n \|x_n\| > 0$.

2. The equivalence (a)$\leftrightarrow$(b) and a preliminary lemma

We begin this section by proving that the equivalence (a)$\leftrightarrow$(b) holds in a quite general situation.

LEMMA 1. *Let* $\{e_n\}$ *be a basis of a Banach space* E, $\|e_n\| = 1$, *and let* $x_n = a_n e_n - b_n e_{n+1}$. *Assume that* $0 < \inf\{|a_n|, |b_n|\} \leqq \sup_n \{|a_n|, |b_n|\} \leqq M < \infty$. *Then* $\{x_n\}$ *is a basis of* $\operatorname{span}\{x_n\}$ *if and only if there exist functionals* $\{f_n\} \subset E^*$ *such that* $\{x_n, f_n\}$ *form a bounded biorthogonal system.*

PROOF. It is well known (cf. [2], p. 68) that if $\{x_n\}$ is a basis of $X = \operatorname{span}\{x_n\}$ then there exist $\{f_n\} \subset X^*$ such that $f_i(x_j) = \delta_{i,j}$ and $\sup_n \|x_n\| \|f_n\| < \infty$. Conversely, let $\{x_n, f_n\}$ be a bounded biorthogonal system with $\sup_n \|f_n\| \leqq M$. Let P_n denote the projection defined by $P_n(\Sigma_1^\infty t_i e_i) = \Sigma_1^n t_i e_i$. It is known (cf. [2], p. 68) that $\sup_n \|P_n\| = K < \infty$. Now, for any $m < n$ and each $\Sigma_{i=1}^n t_i x_i \in X$ we have that

$$\left\|\sum_{i=1}^{m} t_i x_i\right\| \leqq \left\|P_m\left(\sum_{i=1}^{m} t_i x_i\right)\right\| + |t_m b_m| =$$

$$= \left\|P_m\left(\sum_{i=1}^{n} t_i x_i\right)\right\| + \left|b_m f_m\left(\sum_{i=1}^{n} t_i x_i\right)\right| \leqq K\left\|\sum_{i=1}^{n} t_i x_i\right\| +$$

$$+ M^2\left\|\sum_{i=1}^{n} t_i x_i\right\| \leqq (M^2 + K)\left\|\sum_{i=1}^{n} t_i x_i\right\|.$$

This inequality proves that $\{x_n\}$ is a basis in view of ([2], p. 69). The proof is thus complete.

Our next step is to reduce the proof of the Theorem to the case where $\inf_n \{|a_n|, |b_n|\} > 0$. In order to accomplish this we first prove

LEMMA 2. *Let $\{x_n\}$ be an almost diagonal sequence in l_p, $1 < p < \infty$, and let $\{x_n, f_n\}$ satisfy* (a). *Then there exists a constant $d > 0$ such that $\{x_n\}$ is equivalent to the sequence $\{y_n\}$ defined as follows: $y_n = x_n$ if $|a_n|, |b_n| \geqq d$; $y_n = a_n e_n$ if $|b_n| < d$ and $y_n = b_n e_{n+1}$ if $|a_n| < d$.*

PROOF. We may assume without loss of generality that $\inf_n \|x_n\| \geqq 1$. Note first, that if d is small enough and both $\min\{|a_n|, |b_n|\} < d$ and $\min\{|a_{n+1}|, |b_{n+1}|\} < d$ are satisfied then either $|a_{n+1}| < d$ or $|b_n| < d$. Indeed, if both $|a_{n+1}| \geqq d$ and $|b_n| \geqq d$ then $|a_n| < d$ and $|b_{n+1}| < d$; let $f_n = (c_1, c_2, c_3, \cdots) \in l_q = l_p^*$. Then $1 = f_n(x_n) = c_n a_n - c_{n+1} b_n \leqq dM + |c_{n+1}| M$ where $M = \operatorname{Max}\{\sup_n \|f_n\|, \sup_n \|x_n\|\}$. Hence $|c_{n+1}| \geqq M^{-1}(1 - dM)$. On the other hand, $0 = f_n(x_{n+1}) = c_{n+1} a_{n+1} - c_{n+2} b_{n+1}$ and therefore $|c_{n+1}| \leqq |c_{n+2} b_{n+1}| \cdot |a_{n+1}|^{-1} \leqq dM \cdot (1 - d^p)^{-\frac{1}{p}}$. It follows that $M^{-1}(1 - dM) \leqq dM(1 - d^p)^{-\frac{1}{p}}$ which is absurd if $d = d(M)$ is small enough.

In view of this remark and disregarding trivial cases we assume that there is an increasing sequence $\{p(n)\}$ of integers and $d > 0$ such that

(2) $\operatorname{Min}\{|a_k|, |b_k|\} \geqq d$ for all $k \notin \{p(n)\}$

(3) $\operatorname{Min}\{|a_{p(n)}|, |b_{p(n)}|\} < d$ and $\operatorname{Max}\{|a_{p(n)}|, |b_{p(n)}|\} \geqq (1 - d^p)^{\frac{1}{p}}$ for all n.

(4) if $p(n+1) = p(n) + 1$ then either $|b_{p(n)}| < d$ or $|a_{p(n)+1}| < d$. We will show that the sequence $\{y_k\}$ defined by

$$\begin{aligned} y_k &= x_k \quad \text{if} \quad k \notin \{p(n)\} \\ &= a_{p(n)} e_{p(n)} \quad \text{if} \quad k = p(n) \quad \text{and} \quad |b_{p(n)}| < d \\ &= b_{p(n)} e_{p(n)+1} \quad \text{if} \quad k = p(n) \quad \text{and} \quad |a_{p(n)}| < d \end{aligned}$$

is equivalent to $\{x_k\}$. Put $q(n)=p(n)$ if $|b_{p(n)}|<d$ and $q(n)=p(n)-1$ if $|a_{p(n)}|<d$ and let $Y_n=\operatorname{span}\{y_i\}_{i=q(n)+1}^{q(n+1)}$.

It is easy to check that for each $m<n$, $x\in Y_m$ and $y\in Y_n$ we have that

$$\|x+y\|^p=\|x\|^p+\|y\|^p. \tag{5}$$

Also, for each n and any sequence $\{t_i\}_{i=q(n)+1}^{q(n+1)}$ of scalars we have

$$\left\|\sum_{i=q(n)+1}^{q(n+1)} t_i(x_i-y_i)\right\|=(|a_{p(n)}t_{p(n)}|^p+|b_{p(n+1)}t_{p(n+1)}|^p)^{\frac{1}{p}} \tag{6}$$

$$\text{if}\quad q(n)=p(n)-1 \text{ and } q(n+1)=p(n+1)$$

$$=|a_{p(n)}t_{p(n)}| \quad \text{if}\quad q(n)=p(n)-1 \text{ and } q(n+1)=p(n+1)-1$$

$$=0 \quad \text{if}\quad q(n)=p(n) \text{ and } q(n+1)=p(n+1)-1$$

$$=|b_{p(n+1)}t_{p(n+1)+1}| \quad \text{if}\quad q(n)=p(n) \text{ and } q(n+1)=p(n+1).$$

It follows that

$$\left\|\sum_{i=q(n)+1}^{q(n+1)} t_i(x_i-y_i)\right\|\leqq d\,|f_{p(n)}(\Sigma t_ix_i)|$$

$$+\,d\,|f_{p(n+1)}(\Sigma t_ix_i)|\leqq 2dM\left\|\sum_{i=q(n)+1}^{q(n+1)} t_ix_i\right\|;$$

and therefore we get that

$$(1-2DM)\left\|\sum_{i=q(n)+1}^{q(n+1)} t_ix_i\right\|\leqq\left\|\sum_{i=q(n)+1}^{q(n+1)} t_iy_i\right\|\leqq(1+2dM)\left\|\sum_{i=q(n)+1}^{q(n+1)} t_ix_i\right\|. \tag{7}$$

In view of (5), (6) and (7) we have that

$$\left\|\sum_{n=1}^{\infty}\sum_{i=q(n)+1}^{q(n+1)} t_iy_i\right\|^p=\sum_{n=1}^{\infty}\left\|\sum_{i=q(n)+1}^{q(n+1)} t_iy_i\right\|^p,$$

hence $\left\|\sum_{n=1}^{\infty}\sum_{i=q(n)+1}^{q(n+1)} t_ix_i\right\|\leqq\left\|\sum_{n=1}^{\infty}\sum_{i=q(n)+1}^{q(n+1)} t_iy_i\right\|+$

$$+ \left\| \sum_{n=1}^{\infty} \sum_{i=q(n)+1}^{q(n+1)} t_i(x_i - y_i) \right\| \leqq \left\| \sum_{n=1}^{\infty} \sum_{i=q(n)+1}^{q(n+1)} t_i y_i \right\|$$

$$+ \left(\sum_{n=1}^{\infty} \left(2dM \left\| \sum_{i=q(n)+1}^{q(n+1)} t_i x_i \right\| \right)^p \right)^{\frac{1}{p}} \leqq [1 + 2dM(1-2dM)^{-1}].$$

$$\left\| \sum_{n=1}^{\infty} \sum_{i=q(n)+1}^{q(n+1)} t_i y_i \right\|.$$

Similarly, one gets that

$$\left\| \sum_{n=1}^{\infty} \sum_{i=q(n)+1}^{q(n+1)} t_i y_i \right\| \leqq \left\| \sum_{n=1}^{\infty} \sum_{i=q(n)+1}^{q(n+1)} t_i x_i \right\| + (2dM)^2(1-2dM)^{-1} \left\| \sum_{n=1}^{\infty} \sum_{i=q(n)+1}^{q(n+1)} t_i y_i \right\|$$

Assuming that d is small enough these inequalities prove the equivalence of $\{x_n\}$ and $\{y_n\}$. Lemma 2 is thus proved.

Corollary 1. *Let $\{x_n\}$ be a semi-normalized almost diagonal sequence in l_p, $1 < p < \infty$. Then there is a constant $d > 0$ such that $\{x_n\}$ is equivalent to the sequence $\{z_n\}$ defined as follows:*

$$z_n = x_n \quad \text{if} \quad \min\{|a_n|, |b_n|\} \geqq d$$

$$z_n = de_n + b_n e_{n+1} \quad \text{if} \quad |a_n| < d$$

$$z_n = a_n e_n + de_{n+1} \quad \text{if} \quad |b_n| < d.$$

Proof. Define $\{y_n\}$ as in Lemma 2, then $\{x_n\}$ is equivalent to $\{y_n\}$ if d is small enough. Therefore $\{y_n\}$ is also a semi-normalized almost diagonal sequence satisfying (a). Using inequalities similar to (5), (6) and (7), we get that $\{y_n\}$ is equivalent to $\{z_n\}$.

3. Proof of (a) $\Rightarrow$ (c).

In view of Corollary 1 it suffices to consider almost diagonal sequences $\{x_n\}$ with $x_n = a_n e_n - b_n e_{n+1}$ where $0 < \inf_n \{|a_n|, |b_n|\}$. Therefore the proof of the Theorem will be complete if we prove the following:

Proposition. *Let $\{e_n\}$ be the unit vector basis of l_p, $1 < p < \infty$, and let $X_n = e_n - b_n e_{n+1}$ where $\inf |b_n| = d > 0$. Then the following assertions are equivalent:*

(A) $\{x_n\}$ *is equivalent to* $\{e_n\}$.

(B) *there is a sequence* $\{f_n\} \subset l_p^*$ *such that* $\{x_n, f_n\}$ *forms a bounded biorthogonal system.*

(C) *There is a constant* $K > 0$ *such that* $|b_n| \leqq K$ *for all* n *and either*

(C_1) $\sup_n \left(1 + \sum_{k=1}^{n-1} \left| \prod_{i=k}^{n-1} b_i \right|^q \right) \leqq K$

or

(C_2) $\sup_n \left(1 + \sum_{k=1}^{n-1} \left| \prod_{i=k}^{n-1} b_i \right|^q \right) \left(\sum_{j=n}^{\infty} \left| \prod_{i=1}^{j} b_i \right|^{-q} \right) \leqq K$ *where* $p^{-1} + q^{-1} = 1$.

Proof. The implication (A) → (B) is trivial. Let us prove (B) → (C). It is easy to see that either $\operatorname{span}\{x_n\} = l_p$ or $\operatorname{span}\{x_n\}$ is a subspace of codimension 1 of l_p. If $\operatorname{span}\{x_n\} = l_p$ one can easily compute the biorthogonal functionals f_n and verify that $f_n = u_n + \sum_{k=1}^{n-1} (\prod_{i=k}^{n-1} b_i) u_k$ where u_n denotes the n-th unit vector of $l_q = l_p^*$ $(p^{-1} + q^{-1} = 1)$. It follows that

$$1 + \sum_{k=1}^{n-1} \left| \prod_{i=k}^{n-1} b_i \right|^q = \| f_n \|^q \leqq A^q \tag{8}$$

where $A = \sup_n \| f_n \|$. If $\operatorname{span}\{x_n\} \neq l_p$ then there is a functional f which vanishes on $\operatorname{span}\{x_n\}$. It is easy to check that

$$f = u_1 + \sum_{k=2}^{\infty} \left(\prod_{i=1}^{k-1} b_i \right)^{-1} u_k$$

and hence the series

$$\sum_{k=1}^{\infty} \left| \prod_{i=1}^{k} b_i \right|^{-q}$$

converges. Also, for each n,

$$f_n = c_n f + u_n + \sum_{k=1}^{n-1} \left(\prod_{i=k}^{n-1} b_i \right) u_k$$

for some constant c_n. It follows that

$$\left| c_n + \prod_{i=1}^{n-1} b_i \right|^q + \sum_{j=2}^{n-1} \left| \prod_{i=j}^{n-1} b_i + c_n \left(\prod_{i=1}^{j-1} b_i \right)^{-1} \right|^q + \left| 1 + c_n \left(\prod_{i=1}^{n-1} b_i \right)^{-1} \right|^q \tag{9}$$

$$+ \ |c_n|^q \sum_{j=n}^{\infty} \left| \sum_{i=1}^{j} b_i \right|^{-q} = \| f_n \|^q \leqq A^q$$

where $A = \sup_n \| f_n \|$. Put

$$T_k = 1 + \sum_{j=1}^{k-1} \left| \prod_{i=1}^{j} b_i \right|^{-q}, \quad t_k = \sum_{j=k}^{\infty} \left| \prod_{i=1}^{j} b_i \right|^{-q}$$

and

$$S_k = 1 + \sum_{j=1}^{k-1} \left| \prod_{i=j}^{k-1} b_i \right|^{q},$$

then we have that

$$\|f_n\|^q = \left| c_n + \prod_{i=1}^{n-1} b_i \right|^q T_n + |c_n|^q t_n \leqq A^q. \tag{10}$$

One can easily verify that the minimum of the left hand side of (10) is attained at

$$c_n = -\left(\prod_{i=1}^{n-1} b_i\right) T_n^{1/(q-1)} (T_n^{1/(q-1)} + t_n^{1/(q-1)})^{-1}.$$

It follows that

$$\begin{aligned} A^q \geqq \|f_n\|^q &\geqq \left| \prod_{i=1}^{n-1} b_i \right|^q (t^{q/(q-1)} T^q + T^{q/(q-1)} t^q)(T^{1/(q-1)} + t^{1/(q-1)})^{-1} \\ &= \left| \prod_{i=1}^{n-1} b_i \right|^q T_n t_n (T_n^{1/q-1} + t_n^{1/q-1})^{-1}. \end{aligned}$$

Since

$$S_n = \left| \prod_{i=1}^{n-1} b_i \right|^q T_n \text{ and } t_n + T_n = \|f_n\|^q$$

we get that for all $n \geqq 1$

$$K_0 \geqq S_n t_n = \left(1 + \sum_{k=1}^{n-1} \left| \prod_{i=k}^{n-1} b_i \right|^q\right) \left(\sum_{j=n}^{\infty} \left| \prod_{i=1}^{j} b_i \right|^{-q}\right) \tag{11}$$

where $K_0 = 2^{q-1} A^q \|f_n\|^q$. This proves (B) $\rightarrow$ (C). It remains to prove the implication (C) $\rightarrow$ (A). Assume first that (C_1) is satisfied and let T be the operator defined by $Te_n = x_n$. It is clear that $\|T\| \leqq 2K$ and we have to show that T is invertible. Note that $T^* f_n = u_n$ and hence, it suffices to show that the operator $B: l_q \rightarrow l_q$ defined by $Bu_n = f_n$ is bounded. Put $B_i^n = \Pi_{j=i}^n b_j$, then, because $f_n = u_n + \Sigma_{i=1}^{n-1} B_i^{n-1} u_i$, the operator B has the following matrix representation with respect to the basis (u_n):

$$B = \begin{bmatrix} 1 & B_1^1 & B_1^2 & B_1^3 & \cdots & B_1^{n-1} \cdots \\ & 1 & B_2^2 & B_2^3 & & B_2^{n-1} \cdots \\ & & 1 & B_3^3 & & \\ & & & 1 & & \\ & & & & \ddots & \\ & 0 & & & & 1 \\ & & & & & \cdot \end{bmatrix}$$

Let $S_0 = I =$ the identity on l_q and let

$$S_k = \begin{bmatrix} \overbrace{0, 0, \cdots, 0}^{k} & B_1^k & & 0 \\ & & B_2^{k+1} & \\ & 0 & & B_3^{k+2} \end{bmatrix}$$

for $k \geqq 1$. Obviously $\|S_k\| \leqq \max_{i \geqq 1} |B_i^{k+i-1}|$ and $B = \Sigma_{k=0}^{\infty} S_k$. We will prove that $\Sigma_0^{\infty} \|S_k\|$ converges by showing that

(12) $\Sigma_{k=1}^{\infty} \max_{1 \leqq i} |B_i^{k+i-1}|$ is convergent.

It follows from (C_1) that

(13) $K \leqq \Sigma_{i=1}^{n} |B_i^n|^q$ for all n.

Let m be an integer for which

(14) $me^{-q} > K$

and let $i(n) = \max\{1 < j \leqq n : |B_j^n| \leqq e^{-1}\}$.

(13) and (14) imply that

(15) $i(n) \geqq n - m$.

Put $n = i(n,0)$, $i(n) = i(n,1)$ and $i(n,k+1) = i(i(n,k))$ for $k \geqq 1$. We thus have that $i(n,0) \geqq i(n,1) \geqq i(n,2) \geqq \cdots \geqq i(n,s) \geqq 1$ where

(16) $i(n,k) - i(n,k+1) \leqq m$, $i(n,s) - 1 \leqq m$ and $|B_{i(n,k+1)}^{i(n,k)}| \leqq e^{-1}$.

Given $j \leqq n$ we can find k such that $i(n,k+1) < j \leqq i(n,k)$ and so $i(n,k) - j < m$, and $k \geqq E((n-j)/m)$ where $E(r)$ denotes entier r for every real r. Since $B_j^n = (\Pi_{k=0}^{s-1} B_{i(n,k+1)}^{i(n,k)})(\Pi_{i=j}^{i(n,s)} b_i)$ we have that

(17) $|B_j^n| \leqq K^m \exp(-E((n-j)/m))$.

It follows that $\max_{i\geqq 1}|B_i^{k+i-1}| \leqq K^m \exp(-E(k-1)/m))$ hence

$$\|B\| \leqq 1 + K^m \sum_{k=1}^{\infty} \exp(-E((k-1)/m)) = c.$$

This proves that T is invertible and that for any n and any sequence (s_j) of scalars we have that

$$c^{-1}\left\|\sum_{j=1}^{n} s_j e_j\right\| \leqq \left\|\sum_{j=1}^{n} s_j x_j\right\| \leqq 2K\left\|\sum_{j=1}^{n} s_j e_j\right\|. \tag{18}$$

This proves $(C_1) \to (A)$. Now assume (C_2). For each $1 \leqq k \leqq n$ put $\tilde{x}_k = e_{n+2-k} - b_{n+1-k}^{-1}e_{n+1-k} = -b_{n+1-k}^{-1}x_{n+1-k}$ and let $\tilde{f}_1 = u_{n+1}$, $\tilde{f}_2 = u_n + b_n^{-1}u_{n+1}$, and, for $2 \leqq k \leqq n$, let $\tilde{f}_k = u_{n+2-k} + b_{n+2-k}^{-1}u_{n+3-k} + b_{n+2-k}^{-1}b_{n+3-k}^{-1}u_{n+4-k} + \cdots + (\prod_{j=1}^{k-1} b_{n+1-k+j})u_{n+1}$.

It follows from (C_2) that

$$\begin{aligned}\|\tilde{f}_k\|^q &= 1 + |b_{n+2-k}|^{-q} + |b_{n+2-k}b_{n+3-k}|^{-q} + \cdots + \left|\prod_{j=1}^{k-1} b_{n+1-k+j}\right|^{-q} \\ &= \left|\prod_{j=1}^{n+1-k} b_j\right|^q\left(\sum_{i=n+1-k}^{n}\left|\prod_{j=1}^{i} b_j\right|^{-q}\right) \leqq \sup_m \left|\prod_{j=1}^{m} b_j\right|^q\left(\sum_{k=m}^{\infty}\left|\prod_{j=1}^{k} b_j\right|^{-q}\right) \leqq K.\end{aligned} \tag{19}$$

It follows that $(\tilde{x}_k)_{k=1}^n$ is an almost diagonal sequence with respect to the basis $(e_{n+2-k})_{k=1}^{n+1}$ which is a unit vector basis of l_p^{n+1}. Moreover, (C_2) implies that $\{\tilde{x}_k, \tilde{f}_k\}$ is a bounded biorthogonal system such that

$$\|\tilde{f}_k\|^q = 1 + \sum_{j=1}^{n-1}\left|\sum_{i=j}^{n-1} \tilde{b}_i\right|^q \leqq K \text{ where } \tilde{b}_i = b_{n+2-i}^{-1} \text{ and } K^{-1} \leqq \tilde{b}_i \leqq d^{-1}.$$

Hence the argument for $(C_1) \to (A)$ shows that

$$c^{-1}\left\|\sum_{i=1}^{n} s_i e_{n+2-i}\right\| \leqq \left\|\sum_{1}^{n} s_i \tilde{x}_i\right\| \leqq 2d^{-1}\left\|\sum_{1}^{n} s_i e_{n+2-i}\right\|$$

for some constant $c = c(d, K)$ independent on n. Since $\tilde{x}_k = -b_{n+1-k}^{-1} \cdot x_{n+1-k}$

we get that $\{x_n\}$ is also equivalent to $\{e_n\}$. This completes the proof of the Proposition and so the Theorem is fully proven.

4. An application to quadratic forms

Let $R = (r_{i,j})$ be an infinite real symmetric matrix and let $F = F(s_1, s_2, \cdots) = \Sigma_i \Sigma_j r_{ij} s_i s_j$ be the quadratic form wich corresponds to R. F is said to be positive definite if there is a constant $c > 0$ such that

$$F(s_1, s_2, \cdots) \geqq c\left(\sum_{i=1}^{\infty} s_i^2\right) \tag{20}$$

for all sequences of reals $\{s_n\}$. We are interested in the problem, when is the quadratic form $F(s_1, s_2, \cdots) = \Sigma_1^\infty \alpha_i s_i^2 - 2\Sigma_{i=1}^\infty \beta_i s_i s_{i+1}$ which corresponds to the matrix

$$R = \begin{bmatrix} \alpha_1 & -\beta_1 & & & 0 \\ -\beta_1 & \alpha_2 & -\beta_2 & & \\ & -\beta_2 & \alpha_3 & -\beta_3 & \\ 0 & & -\beta_3 & \alpha_4 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

positive definite? It is easy to see that "small" changes in the values of α_i and β_j do not affect the positivity of F, therefore we will state our result as follows:

COROLLARY 2. *Let* $F = F(s_1, s_2, \cdots) = \Sigma_1^\infty \alpha_i s_i^2 - 2\Sigma_{i=1}^\infty \beta_i s_i s_{i+1}$ *with* $0 < d = \inf_n\{|\alpha_n|, |\beta_n|\} \leqq \sup_n\{|\alpha_n|, |\beta_n|\} = M < \infty$. *Then* F *is positive definite if and only if the following conditions are satisfied*

(D_0) *Put* $a_1 = 0$, $b_1 = \alpha_1^{1/2}$ *and for* $n \geqq 1$ *let* $a_{n+1} = \beta_n b_n^{-1}$ *and* $b_{n+1} = (\alpha_{n+1} - a_{n+1}^2)^{\frac{1}{2}}$. *Then there is a positive constant* K *such that* $\alpha_{n+1} - a_{n+1}^2 \geqq K^2$ *and either*

$$\sup_n\left(1 + \sum_{k=2}^{n-1}\left|\prod_{i=k}^{n-1} b_i a_i^{-1}\right|^2\right) < \infty \tag{D_1}$$

or

$$\sup_n\left(1 + \sum_{k=2}^{n-1}\left|\prod_{i=k}^{n-1} b_i a_i^{-1}\right|^2\right)\left(\sum_{k=n}^{\infty}\left|\prod_{i=1}^{k} a_i b_i^{-1}\right|^2\right) < \infty. \tag{D_2}$$

PROOF. Let (D_0) be satisfied, then $b_i a_{i+1} = \beta_i$ and therefore $F(s_1, s_2, \cdots) = \Sigma_1^\infty (a_i^2 + b_i^2) s_i^2 - 2\Sigma b_i a_{i+1} s_i s_{i+1}$. We also have that $0 < \inf_{n \geqq 1}\{|a_{n+1}|, |b_n|\} \leqq \sup_{n \geqq 1}\{|a_n|, |b_n|\} < \infty$ and hence if $x_n = a_n e_n - b_n e_{n+1}$, then, in the l_2-norm $\|\Sigma s_i x_i\|^2 = F(s_1, s_2, \cdots)$. It follows that F is positive definite if and only if the operator $T: l_2 \to \operatorname{span}\{x_n\}$ defined by $Te_i = x_i$ is invertible. We know by the Proposition that T is invertible if and only if either (C_1) or (C_2) is satisfied (with b_i/a_i replacing b_i there). This proves the sufficiency of our condition. Now suppose that F is positive definite. Then putting $s_1 = \beta_1 b_1^{-2}$, $s_2 = 1$ and $s_i = 0$ for $i \geqq 3$ we get by (20) that $(b_1 s_1 - \beta_1 b_1^{-1} s_2)^2 + (\alpha_2 - \beta_1^2 b_1^{-2}) s_2^2 = \alpha_1 s_1^2 + \alpha_2 s_2^2 - 2\beta_1 s_1 s_2 \geqq c(s_1^2 + s_2^2)$ and hence $\alpha_2 - \beta_1^2 b_1^{-2} \geqq c$. Assume that $\alpha_{n+1} - \beta_n^2 b_n^{-2} \geqq c$ for $n = 2, 3, \cdots, k-1$ and proceed by induction. We have that

$$\sum_{i=1}^{k} (b_i s_i - a_{i+1} s_{i+1})^2 + (\alpha_{k+1} - a_{k+1}^2) s_{k+1} = \sum_{i=1}^{k+1} \alpha_i s_i^2 - 2 \sum_{i=1}^{k} \beta_i s_i s_{i+1} \geqq c\left(\sum_{i=1}^{k+1} s_i^2\right)$$

(we put $s_i = 0$ for $i > k+1$).

Now let $s_{k+1} = 1$ and $s_j = a_{j+1} b_j^{-1} s_{j+1}$ for $1 \leqq j \leqq k$. Then we get that $\alpha_{k+1} - a_{k+1}^2 = \alpha_{k+1} - \beta_k^2 b_k^{-2} \geqq c$. It also follows that $0 < \inf_{n \geqq 1}\{|a_{n+1}|, |b_n|\} \leqq \sup_{n \geqq 1}\{|a_n|, |b_n|\} < \infty$ and therefore if $x_n = a_n e_n - b_n e_{n+1}$, then the operator $T: l_2 \to \operatorname{span}\{x_n\}$ defined by $Te_i = x_i$ is invertible and hence, by the Proposition, either (C_1) or (C_2) is satisfied with b_i replaced by $b_i a_i^{-1}$. This proves Corollary 2, which has probably been proved before by using matrix algebra methods.

5. Concluding remarks

(1) In the case of l_2 both the Theorem and the Proposition can be proved by using the Gram-Schmidt orthogonalization of the system $\{x_n\}$. This process results in a system which is very "close" to the biorthogonal system $\{f_n\}$ of $\{x_n\}$. A similar process can be used in connection with Corollary 2; using a Gram-Schmidt orthogonalization of $\{e_n\}$ with respect to the positive definite quadratic form $F = F(x, y)$, we get an orthonormal system $\{z_n\}$ with $z_n = \Sigma_{i=1}^n a_{i,n} e_i$. Define $u: l_2 \to l_2$ by $u(z_n) = e_n$ and put $x_n = u(e_n)$. Then $x_n = \Sigma_{j=1}^n b_{j,n} e_j$ and

$$\begin{aligned}(x_i, x_j) = (ue_i, ue_j) = F(e_i, e_j) &= \beta_{\min(i,j)} && \text{if } |i-j| = 1\\ &= \alpha_i && \text{if } i = j\\ &= 0 && \text{otherwise.}\end{aligned}$$

It is easy to prove by induction that $x_n = a_n e_{n-1} - b_n e_n$ where $a_1 = 0$, $b_n = (\alpha_n - a_n^2)^{1/2}$ and $a_{n+1} = \beta_n b_n^{-1}$. Since u is invertible, $\{x_n\}$ is equivalent to $\{e_n\}$ and hence, by the Proposition, either D_1 or D_2 is satisfied.

(2) The proof of the Theorem is true also in the case of complex l_p spaces.

(3) It seems to us that the proof yields the Theorem also in the case of reflexive Orlicz sequence spaces, however, we did not check this case.

(4) Note that almost diagonal systems have the property that each element has a two-points support (i.e. if $x = (x_1, x_2, x_3, \cdots)$ then $x_i = 0$ for all i except for at most two of them). The following example shows that the Theorem is false if we allow each element x_n to have a three points support: let $\{x_n\}$ be the sequence in l_2 defined by $x_n = e_n - (1/2)^{1/2}(e_{2n+1} + e_{2n+2})$. Let $f_1 = e_1, f_2 = e_2, f_{2n+1} = e_{2n+1} + (1/2)^{1/2} f_n$ and $f_{2n+2} = e_{2n+2} + (1/2)^{1/2} f_n$. Then $(f_i, x_j) = \delta_{i,j}$ and $\|f_n\| \leqq 2^{1/2}$. Hence $\{x_n, f_n\}$ is a bounded biorthogonal system of l_2 but $\{x_n\}$ is not equivalent to $\{e_n\}$. Indeed, let $s_1 = 1$, $s_{n+1} = 2s_n + 1$ and $y_n = 2^{1/2(1-n)} \cdot \sum_{i=s_n}^{s_n + 2^{n-1} - 1} x_i$. Then for each n, $\|\sum_{i=1}^{n} y_i\| \leqq (1 + 2^n \cdot 2^{(1-n)})^{1/2} = 3^{1/2}$.

(5) The problem solved in this paper originated in [3]. It is a special case of the following question: Let $\{e_n\}$ be the unit vector basis of l_p, $1 < p < \infty$, and let $\{p_n\}$ be an incresing sequence of positive integers. Put $x_n = \sum_{j=p_n+1}^{p_{n+2}} a_j e_j$ and assume that $\{x_n\}$ is a semi-normalized basis of span$\{x_n\}$. Is $\{x_n\}$ equivalent to $\{e_n\}$?

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